# The Advection-Diffusion-Equation Form of Faraday's Law and the Fokker-Planck Equation

## *Applications to Magnetic Reconnection and Stellarators*

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# I. Advection-Diffusion Equation & Chaos

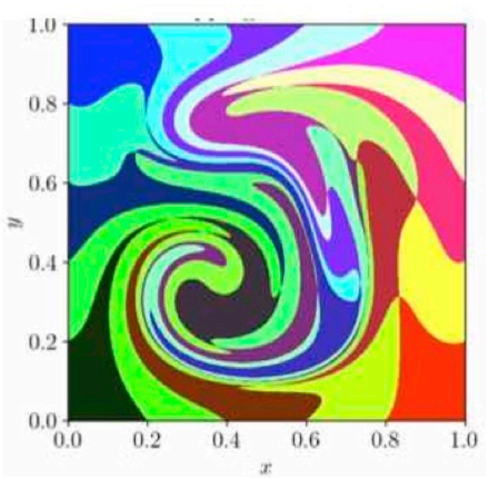
In 1984, H. Aref [1] explained the rapidity of temperature relaxation in a 2D room:

$$\frac{\partial T}{\partial t} + \vec{v}\cdot\vec{\nabla}T = \vec{\nabla}\cdot(D\vec{\nabla}T) \qquad \text{and} \qquad \vec{v} = \hat{z}\times\vec{\nabla}h(x,y,t).$$

**$\vec{v}$ is divergence free;** $\quad v_x \equiv \frac{dx}{dt} = -\frac{\partial h}{\partial y} \quad$ and $\quad v_y \equiv \frac{dy}{dt} = \frac{\partial h}{\partial x}$

***Characteristically the flow $\vec{v}$ is chaotic. A deterministic flow is chaotic in a region when each streamline in that region has a neighboring streamline with a separation that increases exponentially in time.*** *Usual definition of chaos, "a state of complete confusion and lack of order," is not applicable.*

When the diffusion $D = 0$, the area enclosed by a constant-$T$ contour is fixed but the length of the contour increases exponentially with time wherever the flow is chaotic.




*T* was started out in 5X5 square boxes. Within a few evolution times the boxes are contorted by a chaotic flow.

The figure is from Huang and Bhattacharjee, Phys. Plasmas **29**, 122902 (2022) using the chaotic flow of Boozer and Elder, Phys. Plasmas **28**, 062303 (2021).

# Temperature relaxation with a chaotic flow

$$\frac{\partial T}{\partial t} + \vec{v}\cdot\vec{\nabla}T = \vec{\nabla}\cdot(D\vec{\nabla}T) \qquad \text{with} \qquad \vec{\nabla}\cdot\vec{v} = 0.$$

$< T >\equiv \int T dxdy / \int dxdy$ remains constant.

**The entropy-like quantity** $S \equiv -\int T \ln\left(\frac{T}{< T >}\right) dxdy \leq 0$**.**

$dS/dt = \int \frac{D}{T}|\vec{\nabla}T|^2 dxdy \geq 0$ when integrated over a region that encloses the flow.

Wherever $D > 0$, the temperature relaxes to a spatial constant $< T >$.

Chaos causes $|\vec{\nabla}T|^2$ to increase exponentially in time until $D$ relaxes $T$ to a constant.

***For any*** $D > 0$***, chaos causes relaxation in ∼10 times the ideal evolution timescale.***

# II. Magnetic Reconnection [2]

Faraday's Law, $\partial\vec{B}/\partial t = -\vec{\nabla}\times\vec{E}$ implies $\vec{B}$ obeys an advection-diffusion equation.

Simple Ohm's Law example, $\vec{E}+\vec{v}\times\vec{B} = (\eta/\mu_0)\vec{\nabla}\times\vec{B}$, implies

$$\frac{\partial\vec{B}}{\partial t} - \vec{\nabla}\times(\vec{v}_\perp\times\vec{B}) = \frac{\eta}{\mu_0}\nabla^2\vec{B}$$

In 1958 while at LLNL, Bill Newcomb published a proof [3] that when $\eta = 0$ magnetic field lines have a velocity $\vec{v}_\perp$ and cannot change topology.

A subtlety, $\vec{v}_\perp$ can only be chaotic when it is in at least two spatial dimensions and time dependent. For a non-trivial solution, $\vec{v}_\perp\times\vec{B}$ must be non-zero, so $\vec{B}$ must be depend non-trivially on all three spatial coordinates.

***Traditional reconnection theory, as in the 1988 paper of Schindler, Hesse, & Birn [4], rules out chaos, which changes the mathematics fundamentally from that of magnetic reconnection in a general three-dimensional world.***

# Advection-Diffusion Form of Faraday's Law

Anywhere $\vec{B} \neq 0$, the electric field can be written as [5]

$$\vec{E} + \vec{u}_\perp \times \vec{B} = -\vec{\nabla}\Phi + \mathcal{E}\vec{\nabla}\ell$$

with $\ell$ the distance along a line of $\vec{B}$, and $\partial\mathcal{E}/\partial\ell = 0$.

***Proof:*** $\vec{B} \cdot \vec{E} = -\vec{B} \cdot \vec{\nabla}\Phi + \mathcal{E}\vec{B} \cdot \vec{\nabla}\ell$ **or** $\dfrac{\partial\Phi}{\partial\ell} = -E_{||} + \mathcal{E}\hat{b} \cdot \vec{\nabla}\ell$

*This equation can be solved locally for $\Phi$ with $\mathcal{E} = 0$. However, $\mathcal{E}$ must be non-zero if $\Phi$ must solve two boundary conditions in $\ell$, as in a coronal loop, or in a torus.*

*Arbitrary perpendicular components are balanced by $\vec{u}_\perp$, the field line flow speed in an ideal evolution.*

*Line, area, and volume nulls are removed by an arbitrarily small magnetic perturbation. Only point nulls are generic, and they provide a boundary condition on $\Phi$ on an infinitesimal sphere around the point. $\Phi$ must be chosen so charge doesn't accumulate at the null, $\oint \vec{j} \cdot d\vec{a} = 0$.*

$$\frac{\partial\vec{B}}{\partial t} - \vec{\nabla} \times (\vec{u}_\perp \times \vec{B}) = \vec{\nabla}\ell \times \vec{\nabla}\mathcal{E}$$

# General theory of magnetic evolution [2]

A general magnetic evolution consists of three parts when the magnetic Reynolds number $R_m \equiv \frac{\mu_0 u_\perp L}{\eta} >> 1$:

1. **Changes in magnetic field-line topology.** Chaos causes large scale topology breaking and loss of static force balance on a timescale $\tau_t \approx (L/u_\perp)\ln R_m \sim 10(L/u_\perp)$, within current sheets $j \sim (B/\mu_0 L)\ln R_m$.
2. **Dissipation of magnetic energy.** Energy released by breaking field-line topology and loss of force balance first goes into Alfvén waves. Current density rapidly increases until $j \sim (B/\mu_0 L)R_m \propto 1/\eta$ and Alfvén wave dissipation competes. Timescale is $\tau_A \sim 10(L/u_\perp)$ only slightly longer than $\tau_t$.
3. Dissipation of magnetic helicity $K \equiv \int \vec{A}\cdot\vec{B} d^3x$. $K$ dissipation is on a global resistive timescale, $\tau_K \sim \mu_0 L^2/\eta$. Coronal eruptions appear to be due to accumulation of helicity from footpoint twisting.

# III. Plasma Evolution [6]

The plasma evolution obeys the Fokker-Planck equation

$$\frac{\partial f}{\partial t} - \frac{\partial H}{\partial \vec{x}} \cdot \frac{\partial f}{\partial \vec{p}} + \frac{\partial H}{\partial \vec{p}} \cdot \frac{\partial f}{\partial \vec{x}} = C[f],$$

which is an advection-diffusion equation. An oddity of plasmas is the diffusive term gives diffusion in momentum space only.

**Pitch-angle scattering most important** $C_\omega[f] \approx \nu_\omega \frac{mT}{2} \frac{\partial f}{\partial \vec{p}} \cdot (\overleftrightarrow{1} - \hat{p}\hat{p}) \cdot \frac{\partial f}{\partial \vec{p}}$

**Without collisions, entropy per unit volume** $s = -\int f \ln f d^3p$ **is conserved.**

**Pitch-angle scattering gives** $\frac{ds}{dt} \approx \frac{T}{2m} \int \frac{\nu_\omega}{f} \frac{\partial f}{\partial \vec{p}} \cdot (\overleftrightarrow{1} - \hat{p}\hat{p}) \cdot \frac{\partial f}{\partial \vec{p}} d^3p.$

**When $\vec{p}$ is chaotic, entropy creation becomes independent of collisions as $\nu_\omega \to 0$.**

# Traditional non-equilibrium thermodynamics

**Thermodynamic equation** $dU = TdS + \mu_c dN$

**Let** $u = \dfrac{U}{V}, \quad s = \dfrac{S}{V}, \quad n = \dfrac{N}{V}$

$$V(du - Tds - \mu_c dn) + (u - Ts - \mu_c n)dV = 0$$

**When the thermodynamics is independent of system size** $V$**.**

$$du = Tds + \mu_c dn; \quad \text{the chemical potential} \quad \mu_c = \frac{u - Ts}{n}.$$

**Heat and particle fluxes:** $\dfrac{\partial u}{\partial t} + \vec{\nabla} \cdot \vec{Q} = 0, \quad \dfrac{\partial n}{\partial t} + \vec{\nabla} \cdot \vec{\Gamma} = 0$

**Imply** $\dfrac{ds}{dt} \equiv \dfrac{\partial s}{\partial t} + \vec{\nabla} \cdot (\dfrac{1}{T}\vec{Q} - \dfrac{\mu_c}{T}\vec{\Gamma}) = \vec{Q} \cdot \vec{\nabla}\dfrac{1}{T} - \vec{\Gamma} \cdot \vec{\nabla}\dfrac{\mu_c}{T}$

**The thermodynamic forces are** $\vec{\nabla}(1/T)$ **and** $-\vec{\nabla}(\mu_c/T)$**. The entropy flux is** $\vec{\mathcal{F}}_s = (1/T)\vec{Q} - (\mu_c/T)\vec{\Gamma}$**. In linear transport, the fluxes** $Q$ **and** $\Gamma$ **are proportional to the forces.**

# Plasma transport using kinetic theory [7]

The distribution function can always be written as $f = f_M e^{\hat{f}}$, where $f_M$ is a flowing local Maxwellian at each spatial point $\vec{x}$ and $\hat{f}$ gives the deviation, which need not be small. For a single species, $C[f_M] = 0$. $f_M = \frac{n e^{-mv^2/2T}}{(2\pi mT)^{(3/2)}} = e^{\{-1+(\mu_c - mv^2/2)/T\}}$.

When $\int \hat{f}^2 f d^3p << \int f d^3p$, steady-state transport across magnetic surfaces enclosing a toroidal flux $\psi$ is given by entropy creation.

$$-\frac{\partial H}{\partial \vec{x}} \cdot \frac{\partial f}{\partial \vec{p}} + \frac{\partial H}{\partial \vec{p}} \cdot \frac{\partial f}{\partial \vec{x}} = C[f], \text{ or } -\frac{\partial H}{\partial \vec{x}} \cdot \frac{\partial \ln f}{\partial \vec{p}} + \frac{\partial H}{\partial \vec{p}} \cdot \frac{\partial \ln f}{\partial \vec{x}} = \frac{C[f]}{f}, \text{ equivalent to}$$

$$-\frac{\partial H}{\partial \vec{x}} \cdot \frac{\partial \hat{f}}{\partial \vec{p}} + \frac{\partial H}{\partial \vec{p}} \cdot \frac{\partial \hat{f}}{\partial \vec{x}} + \frac{\partial H}{\partial \vec{p}} \cdot \frac{\partial \ln f_m}{\partial \vec{x}} = \frac{C[f]}{f} \equiv C_\ell[\hat{f}],$$

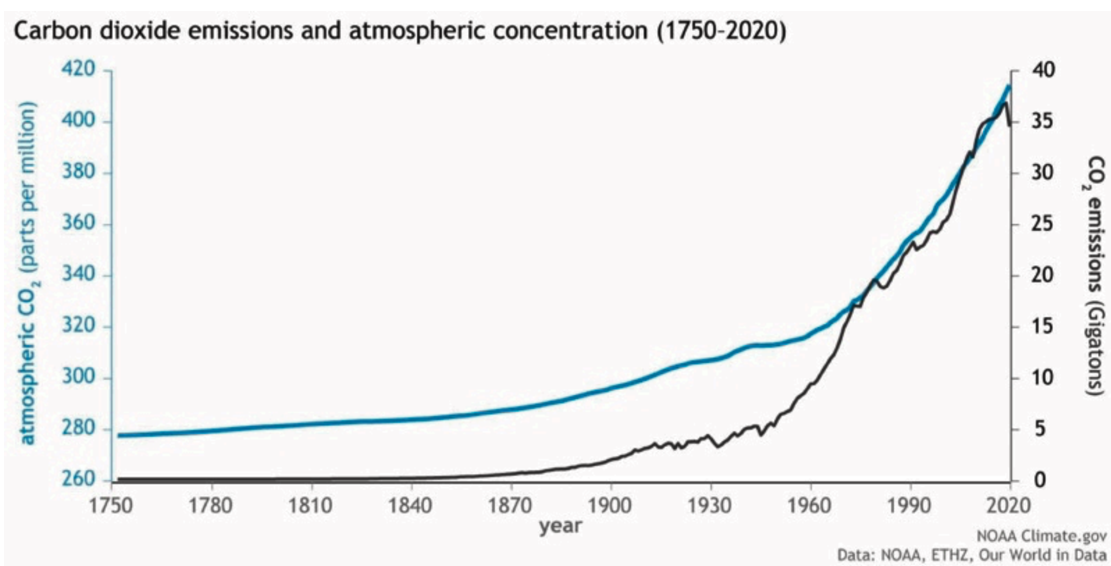
$$\hat{f} \propto \frac{\partial \ln f_M}{\partial \psi} = \frac{\partial(\mu_c/T)}{\partial \psi} - \frac{\partial(mv^2/2T)}{\partial \psi}.$$

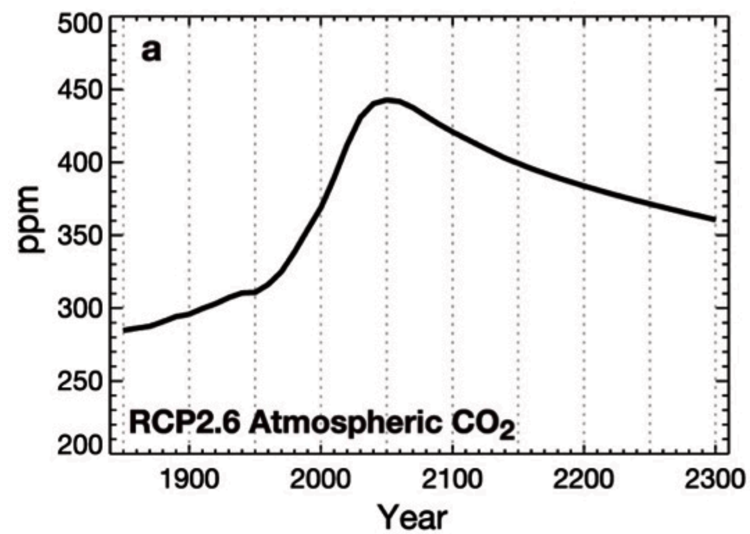
$$\frac{ds}{dt} \approx \frac{T}{2m} \int \nu_\omega f_M \frac{\partial \hat{f}}{\partial \vec{p}} \cdot (\overleftrightarrow{1} - \hat{p}\hat{p}) \cdot \frac{\partial \hat{f}}{\partial \vec{p}} d^3p = Q \frac{d1/T}{d\psi} - \Gamma \frac{d\mu_c/T}{d\psi}.$$

**Gives fluxes $\Gamma$ and $Q$ in term of the forces $-d(\mu_c/T)/d\psi$ and $d(1/T)/d\psi$.**

# IV. Energy Policy & Fusion Energy [8]

Timescale for addressing $CO_2$ problem defined by enhancement doubling time $\sim$30 yrs. Strategy should be defined by deployment costs being $\sim$1000 times development costs. Solution in less than 300 yrs. requires direct removal of $CO_2$. *Natural removal* $\lesssim 4$ *Gt/yr.*



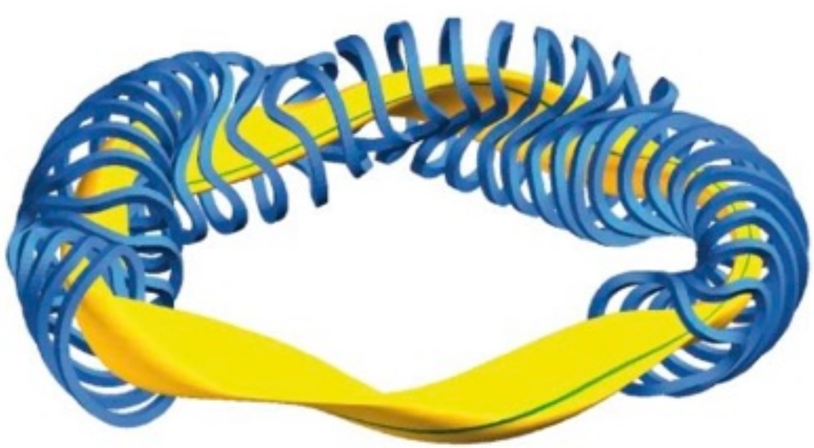

https://www.nap.edu/read/25259/chapter/3#24
Fig. 3a, C D Jones *et al* 2016 *Environ. Res. Lett.* **11** 095012

# Stellarators: Minimal Risk-Time Fusion Path [8, 9]

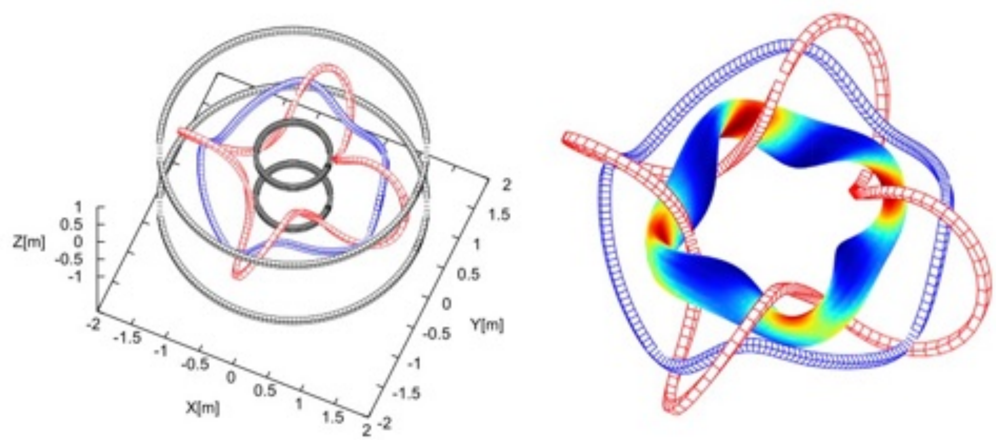
Of all fusion concepts, inertial or magnetic, stellarators have the most complete external control of the plasma.

The external magnetic field defines the plasma configuration to whatever extent seems optimal. Approximately fifty external magnetic field distributions can be efficiently and independently produced—even with open plasma access.

W7-X Stellarator



Yamaguchi, Nucl. Fusion **59**, 104002 (2019)



# Why stellarator development delayed relative tokamaks

In toroidal fusion plasmas, $\vec{\nabla} p = \vec{j} \times \vec{B}$, constant-pressure surfaces coincide with magnetic surfaces, $\vec{B} \cdot \vec{\nabla} p = 0$.

Appropriate nested magnetic surfaces are not possible unless there is a strong plasma current in axisymmetric tokamaks, but can be produced when the magnetic field has an approximate 20 % periodic variation toroidally.

In the early 1980's, this variation was thought to exclude adequate particle confinement for fusion.

The strong plasma current in tokamaks was known even then to allow disruptions, a sudden loss plasma confinement, which would cause unacceptable damage to a power plant.

**Can be solved by using a stellarator.**

# Elimination of stellarator fatal flaw

In 1984, I showed that small-gyroradius particle drift motion is given by a simple a Hamiltonian in a coordinate system that I developed [10]. When the field strength in this coordinate system depends on the poloidal $\theta$ and the toroidal angle $\varphi$ in a fixed linear combination $\theta + N\varphi$ a component of the canonical momentum is conserved. This gives quasi-symmetry—axisymmetric-like particle transport [11].

In 1988, Nührenberg and Zille [12] showed coils can produce a field strength consistent with quasi-helical symmetry, $B(\psi, \theta + N\varphi)$; toroidal flux enclosed by a surface is $\psi$.

In late 1980's, Nührenberg [13] used the drift Hamiltonian and confinement concept of omnigeneity developed by Hall and McNamara [14] at LLNL in 1975 to design W7-X.

## W7-X has demonstrated:

1. Stellarators can have adequate particle confinement for fusion when appropriately designed,
2. Computational design is highly reliable for stellarators even with radical changes because the physics is dominated by the coil-produced magnetic field.

# Stellarator confinement requires elimination of large-spatial-scale chaos

When the parallel gyroradius, $\rho_{||} \equiv \frac{v_{||}}{\omega_c}$, is small compared to the spatial variation along $\vec{B}$ in time-independent problems, the magnetic moment $\mu \equiv \frac{mv_\perp^2}{2B}$ is conserved.

The particle energy $H = \frac{1}{2}mv_{||}^2 + \mu B + q\Phi$, and the motion of the center of its circular motion about $\vec{B}$ is $\vec{v}_g = \frac{v_{||}}{B}\vec{B}_{eff}$, where $\vec{B}_{eff} \equiv \vec{B} + \vec{\nabla}\left(\rho_{||}(H, \mu, v_{||}, \vec{x})\vec{B}\right)$ as I showed in 1980 [15]. The electric potential is usually almost constant along $\vec{B}$,

**Two types of large-spatial-scale chaos:**

1. Magnetic field line chaos for passing particles with $H > \mu B_{max} + q\Phi$, which closely follow $\vec{B}_{eff}$ lines.

2. Non-conservation of longitudinal action [16] $\oint v_{||} d\ell$ for trapped particles.

Passing particle confinement is easier to obtain than trapped. Even when $\oint v_{||} d\ell$ is adiabatically conserved the particle drifts can carry particles too far across magnetic surfaces unless the trapping wells have a certain symmetry, which gives Hall and McNamara's omnigeneity [6, 14].

# Gyro-Scale Chaos gives Gyro-Bohm Transport

Quasi-neutrality requires the electric potential must scale as $\Phi \sim T/e$.

Microturbulence arises because of the different response of ions and electrons to an electric perturbation across the magnetic field lines.

A natural spatial scale for perturbations is $\xi \sim \rho_s$, where $\rho_s$ is the ion gyroradius calculated with the speed of sound; $\rho_s \equiv m_i C_s/eB$.

This displacement creates an electric field $E \sim T_d/e\xi$ and a drift $\vec{E} \times \vec{B}/B^2 \sim T_d/(e\xi B)$, where $T_d$ is the temperature difference in the plasma on the spatial scale $\xi$. This implies $T_d \sim (\xi/a)T$, where $a$ is the thermal scale length, which in the plasma interior can be approximated by the minor radius.

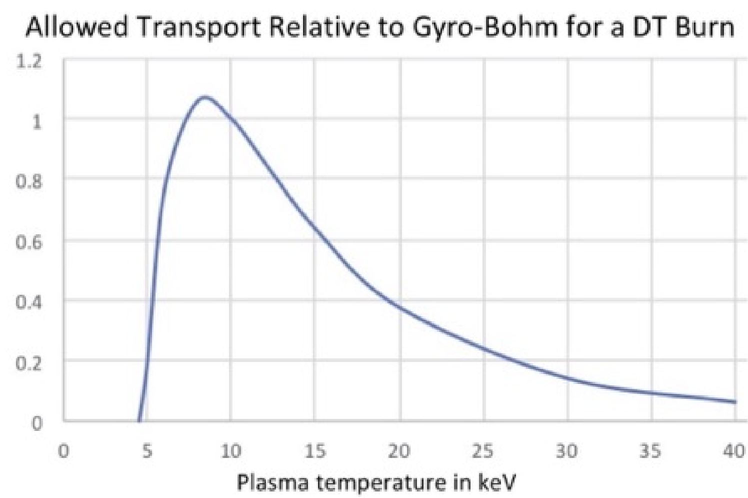
The characteristic diffusion coefficient is then $D \sim \xi^2/\tau_c$, where the characteristic time scale $\tau_c = \xi/(\vec{E} \times \vec{B}/B^2) \sim \xi^2 eB/T_d$. Consequently [17], the diffusion coefficient has a Bohm-like form, $D \sim T_d/eB$, so $D \sim \frac{\xi}{a}\frac{T}{eB}$ with $\xi$ typically $\sim \rho_s$

**Empirical transport is generally gyro-Bohm** $D \sim \dfrac{\rho_s}{a}\dfrac{T}{eB}$.

# Ignition with gyro-Bohm transport

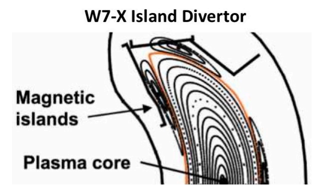
Gyro-Bohm transport is consistent with attractive fusion power plants [9] that operate at ≈10 keV. But, a DT burn is an order of magnitude more difficult [6] to achieve at 35 keV.

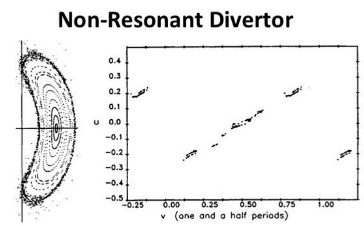
Stellarator power plant designs have a typical temperature of 10 keV, but tokamak power plant designs frequently have ≈35 keV in order to efficiently maintain the current and to have an adequate power density with a density below the Greenwald limit.



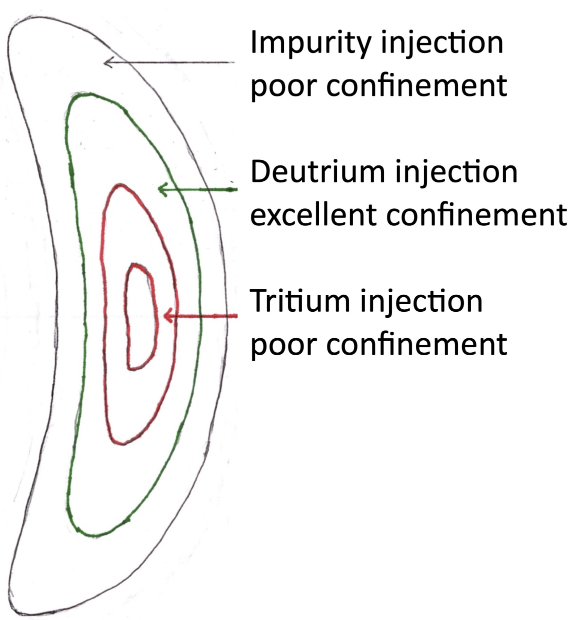
# Divertors, Sparing Tritium, and Impurity Control

Need to pump out helium produced by DT fusion, but should radiate energy to spread it over the walls [17]. Three-layer confinement, which is natural to stellarators [18], should aid this.




Neutron damage gives a material limit of $\approx 10$ MW yr/m$^2$. Makes easy access to plasma chamber critical for cheapness of fusion power, 80% of power comes out in neutrons. *The helical stripes in non-resonant divertors show trajectory collimation in chaos.*



# Summary

Temperature equilibration in a room requires diffusion but takes only an order magnitude longer than the timescale of the ideal flow. Chaos in the flow is essential, an exponential increase in time of the separation of neighboring streamlines.

Faraday's Law and the Fokker-Planck equation are of the advection-diffusion form. In three-dimensions, (1) large-scale breaking of field-line connections takes $\sim 10$ times the ideal evolution time, (2) released magnetic energy is rapidly damped via Alfvén wave damping, and (3) boundary conditions can insert magnetic helicity, but its dissipation is on the global resistive timescale.

The Fokker-Planck equation implies that without collisions there can be no change in the entropy per unit volume, but chaos in the particle trajectories implies that entropy can be created at a rate that is essentially independent of the collision frequency. Chaotic trajectories are only consistent with fusion power when on a small spatial scale.

W7-X has demonstrated that large-scale chaos can be eliminated in stellarators. They appear to offer the minimum time and risk path to fusion power—largely due to the external control of the plasma.

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